PARAMETER MISMATCHES, VARIABLE DELAY TIMES AND SYNCHRONIZATION IN TIME-DELAYED SYSTEMS

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ABSTRACT

We investigate synchronization between two unidirectionally linearly coupled chaotic non-identical time-delayed systems and show that parameter mismatches are of crucial importance to achieve synchronization. We establish that independent of the relation between the delay time in the coupled systems and the coupling delay time, only retarded synchronization with the coupling delay time is obtained. We show that with parameter mismatch or without it neither complete nor anticipating synchronization occurs. We derive existence and stability conditions for the retarded synchronization manifold. We demonstrate our approach using examples of the Ikeda and Mackey Glass models. Also for the first time we investigate chaos synchronization in time-delayed systems with variable delay time and find both existence and sufficient stability conditions for the retarded synchronization manifold with the coupling-delay lag time. Also for the first time we consider synchronization between two unidirectionally coupled chaotic multi-feedback Ikeda systems and derive existence and stability conditions for the different anticipating, lag, and complete synchronization regimes.

Key word(s):chaos synchronization, time-delayed systems, variable delay times, parameter mismatches, multi-feedback systems

1.INTRODUCTION

Seminal papers on chaos synchronization [1] have stimulated a wide range of research activity especially extensively in lasers, electronic circuits, chemical and biological systems [2]. Possible application areas of chaos synchronization are in secure communications, optimization of nonlinear system performance, modeling brain activity and pattern recognition phenomena [2].

There are different types of sychronization in interacting chaotic systems. Complete, generalized, phase, lag and anticipating synchronizations of chaotic oscillators have been described theoretically and observed experimentally. Complete synchronization implies coincidence of states

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of interacting systems, $y(t) = x(t)$ [1]. Generalized synchronization is defined as the presence of some functional relation between the states of response and drive, i.e. $y(t) = F(x(t))$ [3]. Phase synchronization means entrainment of phases of chaotic oscillators, $n\Phi_x - m\Phi_y = const$, ($n$ and m are integers) whereas their amplitudes remain chaotic and uncorrelated [4]. Lag synchronization *for the first time* was introduced by Rosenblum *et al.* [5] under certain approximations in studying synchronization between *bi-directionally* coupled systems described by the ordinary differential equations (no intrinsic delay terms) with *parameter mismatches*:$y(t) \approx x_\tau(t) \equiv x(t-\tau)$ with positive τ. Anticipating synchronization [6-8] also appears as a coincidence of shifted-in-time states of two coupled systems, but in this case the driven system anticipates the driver, $y(t) = x(t+\tau)$ or $x = y_\tau, \tau > 0$. An experimental observation of anticipating synchronization in external cavity laser diodes [9] has been reported recently, see also [10] for the theoretical interpretation of the experimental results. The concept of inverse anticipating synchronization $x = -y_\tau$ is introduced in [11].

Due to finite signal transmission times, switching speeds and memory effects time-delayed systems are ubiquitous in nature, technology and society [12]. Therefore the study of synchronization phenomena in such systems is of high practical importance. Time-delayed systems are also interesting because the dimension of their chaotic dynamics can be made arbitrarily large by increasing their delay time. From this point of view these systems are especially appealing for secure communication schemes [13].

Role of parameter mismatches in synchronization phenomena is quite versatile. In certain cases parameter mismatches are detrimental to the synchronization quality: in the case of small parameter mismatches the synchronization error does not decay to zero with time, but can show small fluctuations about zero or even a non-zero mean value; larger values of parameter mismatches can result in the loss of synchronization [8,14]. In some cases parameter mismatches change the time shift between the synchronized systems [15]. In certain cases their presence is necessary for synchronization. We reiterate that the crucial role of parameter mismatches for lag synchronization between *bi-directionally* coupled systems was first studied in [5] by Rosenblum *et al.*. As such, lag synchronization cannot be observed if two oscillators are completely identical, see e.g. [16] and references therein.

Multi-feedback and multi-delay systems are ubiquitous in nature and technology. Prominent examples can be found in biological and biomedical systems,laser physics, integrated communications [12]. In laser physics such a situation arises in lasers subject to two or more optical or elctro-optical feedback. Second optical feedback could be useful to stabilize laser intensity [17]. Chaotic behaviour of laser systems with two optical feedback mechanism is studied in recent works [18]. To the best of our knowledge chaos synchronization between the multi-feedback systems is to be investigated yet. Having in mind enormous application implications of chaos synchronization e.g. in secure communication, investigation of synchronization regimes (lag, complete, anticipating etc.) in multi-feedback systems is of immense importance.

In this paper we investigate synchronization between the two *unidirectionally* coupled chaotic

non-identical time-delayed systems having a fairly general form of coupling and show for the first time that parameter mismatches are, in fact, of crucial importance for achieving synchronization. We show that independent of the relation between the delay time in the coupled systems and the coupling delay time, only retarded (lag) synchronization is obtained. (Usually for lag synchronization between the unidirecitionally coupled time-delayed systems the term retarded synchronization is prefered [8].) In this case the lag time is the coupling delay time. We consider both constant and variable feedback delay times. We demonstrate our approach using examples of the Ikeda and Mackey Glass models.

In the paper also for the first time we investigate synchronization between two unidirectionally coupled chaotic multi-feedback Ikeda systems and find both existence and stability conditions for different synchronization regimes.

2.GENERAL THEORY

Consider a situation where a time-delayed chaotic master (driver) system

$$\frac{dx}{dt} = -\alpha_1 x + k_1 f(x_{\tau_1}), \tag{1}$$

drives a non-identical slave (response) system

$$\frac{dy}{dt} = -\alpha_2 y + k_2 f(y_{\tau_1}) + k_3 x_{\tau_2}, \tag{2}$$

where x and y are dynamical variables; $f(x)$ is differentiable nonlinear function; α_1 and α_2 are relaxation coefficients for the driving and driven dynamical variables, respectively:throughout the paper we assume that $\alpha_1 = \alpha - \delta$ and $\alpha_2 = \alpha + \delta$, δ determines the mismatch of relaxation coefficents; τ_1 is the feedback delay time in the coupled systems;τ_2 is the coupling delay time between the systems. k_1 and k_2 are the feedback rates for the master and the response systems, respectively;k_3 is the linear coupling rate between the driver and the response system.
Now we will show that chaotic systems (1) and (2) can be synchronized on the retarded synchronization manifold with the lag time τ_2:

$$y = x_{\tau_2}. \tag{3}$$

We denote the error signal by $\Delta = x_{\tau_2} - y$. Then from systems (1) and (2) we find the following error dynamics: $\frac{d\Delta}{dt} = -\alpha_2\Delta + (2\delta - k_3)x_{\tau_2} + k_1 f(x_{\tau_1+\tau_2}) - k_2 f(y_{\tau_1})$. Thus under conditions

$$2\delta = k_3, k_1 = k_2, \tag{4}$$

the error dynamics can be written as:

$$\frac{d\Delta}{dt} = -\alpha_2\Delta + k_1\Delta_{\tau_1} f'(x_{\tau_1+\tau_2}). \tag{5}$$

It is obvious that $\Delta = 0$ is a solution of system (5). To study the sufficient stability condition for the retarded synchronization manifold $y = x_{\tau_2}$ one can use a Krasovskii-Lyapunov functional approach [12, 19].
The sufficient stability condition for the trivial solution $\Delta = 0$ of eq.(5) can be found by investigating the positively defined Krasovskii-Lyapunov fuctional

$$V(t) = \frac{1}{2}\Delta^2 + \mu \int_{-\tau}^{0} \Delta^2(t + t_1)dt_1, \tag{6}$$

where $\mu > 0$ is an arbitrary positive parameter. According to [12,19], the solution $\Delta = 0$ is stable, if the derivative of the functional (6) along the trajectory of equation $\frac{d\Delta}{dt} = -r(t)\Delta - s(t)\Delta_\tau$ is negative. In general this negativity condition is of the form: $4(r-\mu)\mu > s^2$ and $r > \mu > 0$. As the value of μ that will allow s^2 as large as possible is $\mu = \frac{r}{2}$, the asymptotic stability condition for $\Delta = 0$ can be written as

$$r^2 > s^2, \tag{7}$$

which is equivalent to $r > |s|$. This result is valid for both constant and time-dependent coefficients r and s (in the latter case $r(t)$ and $s(t)$ should be bounded continuous functions [12]). Thus we obtain that

$$\alpha_2 > |k_1 f'(x_{\tau_1+\tau_2})| \tag{8}$$

is the sufficient stability condition for retarded synchronization manifold (3). The condition (4) is the existence condition of retarded synchronization between the unidirectionally coupled systems (1) and (2).
Thus we find that under certain conditions systems (1) and (2) admit the retarded chaos synchronization manifold $y = x_{\tau_2}$ *only* under parameter mismatch ie $\alpha_1 \neq \alpha_2$. We also notice that without the parameter mismatch, i.e. $\alpha_1 = \alpha_2 = \alpha$ neither $y = x_{\tau_2-\tau_1}$ nor $y = x_{\tau_1-\tau_2}$ is the synchronization manifold. We also emphasize that, in general for both $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_2$ systems (1) and (2) admits neither complete nor anticipating *chaos* synchronization.

So far we have considered the case of constant feedback delay time τ_1. It is of immense interest to study chaos synchronization in time-delayed systems with variable feedback delay time. Basic interest is driven by the fact that so far there are no reported research on this particular subject in the literature. Practical interest is motivated by the appreciation that time-delayed systems with variable delay times are more realistic. As an example one can refer to the biological biorhythms, where the capacity of assimilation of nutrients by an organism varies cyclicly during the day [20] . Now we will try to find both the existence and stability conditions for the synchronization manifold (3) in the case of variable feedback delay times. It is straightforward to establish that the analog of the error dynamics equation in the case of variable delay time $\tau_1(t)$ is of the form:

$$\frac{d\Delta}{dt} = -\alpha_2\Delta + k_1\Delta_{\tau_1(t)} f'(x_{\tau_1(t)+\tau_2}). \tag{9}$$

Again, as in the case of constant feedback delay times equation (9) is obtained from the studying the coevolution of eqs.(1) and (2) along the manifold (3). Analysis of the error dynamics shows that the existence conditions (4) hold for the variable delay cases. Next let us find the sufficient stability condition for system (9). According to [12] for that purpose one can still use the functional (6). Namely as presented in [12], when $\tau = \tau(t)$ is continuously differentiable and bounded, the solution $\Delta = 0$ to $\frac{d\Delta}{dt} = -r(t)\Delta - s(t)\Delta_{\tau(t)}$ is uniformly asymptotically stable, if $a(t) > \mu > 0$ and $(2r(t)-\mu)(1-\frac{d\tau}{dt})\mu > s^2(t)$ uniformly in t. Applying the same procedure as in the case of constant feedback delay time, we can find the value of μ that will allow s^2 to be as large as possible: $\mu = r$. Thus we find that the sufficent stability condition for the $\Delta = 0$ solution of time delay equation with time dependent coefficients $\frac{d\Delta}{dt} = -r(t)\Delta - s(t)\Delta_{\tau(t)}$ is:

$$r^2(t)(1-\frac{d\tau(t)}{dt}) > s^2(t). \tag{10}$$

Notice that for the constant delay time cases the inequality (10) is reduced to the well-known sufficent stability condition $r > |s|$.
As in our case $r = \alpha_2$ and $s = -k_1 f'(x_{\tau_1(t)+\tau_2})$ then the sufficent stability condition for synchronization manifold (3) for the time-delayed equations (1) and (2) with time dependent feedback delay τ_1 can be written as:

$$\alpha_2^2(1-\frac{d\tau_1(t)}{dt}) > (k_1 f'(x_{\tau_1(t)+\tau_2}))^2. \tag{11}$$

3.1.EXAMPLE 1:THE IKEDA MODEL

In this subsection we demonstrate our general theory using the example of the Ikeda model. This investigation is of considerable practical importance, as the equations of the class B lasers with feedback (typical representatives of class B are solid-state, semiconductor, and low pressure CO_2 lasers [21]) can be reduced to an equation of the Ikeda type [22]. Consider synchronization between the Ikeda systems [6],

$$\frac{dx}{dt} = -\alpha_1 x - \beta \sin x_{\tau_1},$$

$$\frac{dy}{dt} = -\alpha_2 y - \beta \sin y_{\tau_1} + K x_{\tau_2}. \tag{12}$$

The Ikeda model was introduced to describe the dynamics of an optical bistable resonator and is well-known for delay-induced chaotic behavior [23]. Physically x is the phase lag of the electric field across the resonator; α is the relaxation coefficient; β is the laser intensity injected into the system. τ_1 is the round trip time of the light in the resonator or feedback delay time in the coupled systems;τ_2 is the coupling delay time between systems x and y.
First we consider the case of constant feedback delay time and show that $y = x_{\tau_2}$ is the retarded

synchronization manifold, if the parameter mismatch $\alpha_2 - \alpha_1 = 2\delta$ is equal to the coupling rate K. This can be seen by the dynamics of the error $\Delta = x_{\tau_2} - y$:

$$\frac{d\Delta}{dt} = -(\alpha + \delta)\Delta + (2\delta - K)x_{\tau_2} - \beta \cos x_{\tau_1+\tau_2}\Delta_{\tau_1}. \tag{13}$$

(As in this example under study we choose feedback rates (β) equal for both the driver and driven systems, the second of the existence conditions in (4) becomes redundant.) The sufficient stability condition for the retarded synchronization manifold $y = x_{\tau_2}$ can be written as:$\alpha + \delta = \alpha_2 > |\beta|$. Thus, as in case of general approach, we find that the retarded chaos synchronization manifold $y = x_{\tau_2}$ occurs *only* under parameter mismatch ie $\alpha_1 \neq \alpha_2$. By analyzing the corresponding error dynamics one can also establish that without the parameter mismatch, i.e. $\alpha_1 = \alpha_2 = \alpha$ neither $y = x_{\tau_2-\tau_1}$ nor $y = x_{\tau_1-\tau_2}$ is the synchronization manifold. We also emphasize that for both $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_2$ system (12) admits neither complete (we notice that for special case of $\tau_2 = 0$ $y = x_{\tau_2}$ is the complete synchronization manifold, which exists if $\alpha_1 \neq \alpha_2$) nor anticipating *chaos* synchronization. We emphasize that this result is due to the linear coupling between the synchronized systems. The importance of the role of the form of coupling between the synchronized systems is underlined in [6,24]. In the case of nonlinear (sinusoidal) coupling for identical drive and response Ikeda systems, depending on the relation between the feedback delay time and the coupling delay time retarded, complete or anticipating synchronization can occur, see, e.g. [25] and references therein.
According to estimations the parameters values $\alpha_1 = 5$, $\alpha_2 = 25$, $\beta = 20$, $K = 20$ and $\tau_1 = 1$, $\tau_2 = 2$ (or $\tau_1 = 3$, $\tau_2 = 1$) satisfy both existence and stability conditions for the retarded chaos synchronizartion manifold $y = x_{\tau_2}$ for the coupled systems (12).

Next we consider the case of time dependent delay time $\tau_1(t)$. First we notice that as in the case of time-independent delay times $2\delta = K$ is the condition of existence for the $y = x_{\tau_2}$ synchronization manifold. Next applying the general formula (11) derived earlier in the paper we write the sufficent stability condition for the synchronization manifold $y = x_{\tau_2}$ in the following form:

$$\alpha_2^2(1 - \frac{d\tau_1(t)}{dt}) > \beta^2, \tag{14}$$

As an example consider the following sinusoidal form of the variable delay time :

$$\tau_1(t) = \tau_0 + \tau_a \sin(\omega t), \tag{15}$$

where τ_0 is the zero frequency component;τ_a is the amplitude; $\frac{\omega}{2\pi}$ is the frequency of the modulation. Then for the concrete form of variable delay time (15) the sufficient stability condition (14) can be writen as:

$$\alpha_2^2(1 - \tau_a\omega\cos(\omega t)) > \beta^2. \tag{16}$$

3.2.EXAMPLE 2:THE MACKEY GLASS MODEL

In this subsection we demonstrate our approach using the example of the Mackey Glass model. The Mackey Glass model has been introduced as a model of blood generation for patients with leukemia and nowadays is very popular in chaos theory [19].

Consider synchronization between the Mackey Glass systems

$$\frac{dx}{dt} = -\alpha_1 x + k_1 \frac{a_1 x_{\tau_1}}{1 + x_{\tau_1}^b},$$

$$\frac{dy}{dt} = -\alpha_2 y + k_2 \frac{a_2 x_{\tau_1}}{1 + x_{\tau_1}^b} + k_3 x_{\tau_2}. \tag{17}$$

The dynamical variable in the Mackey Glass model is the concentration of the mature cells in blood at time t and the delay time is the time between the initiation of cellular production in the bone marrow and the release of mature cells into the blood [20].

Again by investigating the corresponding error dynamics we can show that $y = x_{\tau_2}$ is the retarded synchronization manifold, if the parameter mismatch $\alpha_2 - \alpha_1 = 2\delta$ is equal to the coupling rate k_3 and $k_1 a_1 = k_2 a_2$. We notice that here we can allow for parameter mismatches for a, and thus have more flexibility to achieve synchronization. With these existence conditions, the sufficient stability condition for the retarded synchronization manifold $y = x_{\tau_2}$ can be written as: $\alpha_2 > |k_1 a_1 f'(x_{\tau_1+\tau_2})|$, with $f(x_\tau) = \frac{x_\tau}{1+x_\tau^b}$.

For analytical estimation of α_2 we take into account that the absolute maximum of the function $|f'(x_\tau)|$ is obtained at $x_\tau = (\frac{b+1}{b-1})^{\frac{1}{b}}$ and is equal to $\frac{(b-1)^2}{4b}$ [19]. Thus we arrive at the following sufficient stability condition for the synchronization manifold $y = x_{\tau_2}$ for the coupled systems (17) :

$$\alpha_2 > k_1 a_1 \frac{(b-1)^2}{4b}. \tag{18}$$

Again we would like to underline that only *retarded* synchronization occurs notwithstanding the relation between the feedback delay time and coupling delay time; moreover for both $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_2$ coupled systems (17) admit neither complete nor anticipating *chaos* synchronization.

According to estimations the parameters values $\alpha_1 = 0.1$, $\alpha_2 = 5$, $k_1 a_1 = 2$,$b = 10$, $k_3 = 4.9$,$\tau_1 = 10$ and $\tau_2 = 20$ (or $\tau_1 = 50$ and $\tau_2 = 20$, $\tau_1 = \tau_2 = 50$) satisfy both existence and stability conditions for the retarded chaos synchronizartion manifold $y = x_{\tau_2}$ for the coupled systems (17).

4.SYNCHRONIZATION BETWEEN THE MULTI-FEEDBACK IKEDA SYSTEMS

Consider synchronization between the multi-feedback Ikeda systems,

$$\frac{dx}{dt} = -\alpha x + m_1 \sin x_{\tau_1} + m_2 \sin x_{\tau_2}, \tag{19}$$

$$\frac{dy}{dt} = -\alpha y + m_3 \sin y_{\tau_1} + m_4 \sin y_{\tau_2} + K \sin x_{\tau_3}, \tag{20}$$

with positive $\alpha_{1,2}$ and $-m_{1,2,3,4}$.
As mentioned above physically x is the phase lag of the electric field across the resonator; α is the relaxation coefficient for the driving x and driven y dynamical variables;$-m_{1,2}$ and $-m_{3,4}$ are the laser intensities injected into the driving and driven systems,respectively. $\tau_{1,2}$ are the feedback delay times in the coupled systems;τ_3 is the coupling delay time between systems x and y;K is the coupling rate between the driver x and the response system y.
First we will show that systems (19) and (20) can be synchronized on the manifold:

$$y = x_{\tau_3 - \tau_1}. \tag{21}$$

(For $\tau_3 > \tau_1$,$\tau_3 = \tau_1$, and $\tau_3 < \tau_1$ (21) is the retarded, complete, and anticipating synchronization manifold, respectively.) We denote the error signal by $\Delta = x_{\tau_3-\tau_1} - y$. Then from systems (19) and (20)we find the following error dynamics $\Delta = x_{\tau_3-\tau_1} - y$:$\frac{d\Delta}{dt} = -\alpha\Delta + ((m_1 - K)\sin x_{\tau_3} - m_3 \sin y_{\tau_1}) + m_2 \sin x_{\tau_2+\tau_3-\tau_1} - m_4 \sin y_{\tau_2}$ Thus under conditions

$$m_1 - K = m_3, m_2 = m_4 \tag{22}$$

the error dynamics can be written as:

$$\frac{d\Delta}{dt} = -\alpha\Delta + m_3\Delta_{\tau_1} \cos x_{\tau_3} + m_2\Delta_{\tau_2} \cos x_{\tau_2+\tau_3-\tau_1}. \tag{23}$$

It is obvious that $\Delta = 0$ is the solution of system (23).To study the stability of the synchronization manifold $y = x_{\tau_3-\tau_1}$ one can again use a Krasovskii-Lyapunov functional approach. According to [12], the sufficient stability condition for the trivial solution $\Delta = 0$ of time-delayed equation $\frac{d\Delta}{dt} = -r(t)\Delta + s_1(t)\Delta_{\tau_1} + s_2(t)\Delta_{\tau_2}$ is: $r(t) > |s_1(t)| + |s_2(t)|$.
Thus we obtain that the sufficient stability condition for the synchronization manifold $y = x_{\tau_3-\tau_1}$ (21) can be written as:

$$\alpha > |m_3| + |m_2|. \tag{24}$$

Conditions (22) are the existence conditions for the synchronization manifold (21) between the unidirectionally coupled multi-feedback systems (19) and (20).
Analogously we find that $y = x_{\tau_3-\tau_2}$ is the synchronization manifold between systems (19) and (20) with corresponding existence $m_2 - K = m_4$ and $m_1 = m_3$ and stability conditions $\alpha > |m_3| + |m_4|$.

One can easily generalize the previous rezults to n-tuple feedback Ikeda systems. Indeed consider synchronization between the following Ikeda models:

$$\frac{dx}{dt} = -\alpha x + m_{1x} \sin x_{\tau_1} + m_{2x} \sin x_{\tau_2} + \cdots + m_{nx} \sin x_{\tau_n}, \tag{25}$$

$$\frac{dy}{dt} = -\alpha y + m_{1y} \sin y_{\tau_1} + m_{2y} \sin y_{\tau_2} + m_{ny} \sin y_{\tau_n} + k \sin x_{\tau_k}, \tag{26}$$

Then the existence and sufficent stability conditions e.g. for the synchronization manifold $y = x_{\tau_k - \tau_1}$ are:$m_{1x} - k = m_{1y}, m_{nx} = m_{ny}$ and $\alpha > |m_{1y}| + |m_{2y}| + \cdots + |m_{ny}|$, respectively. For synchronization manifold $y = x_{\tau_k - \tau_2}$, $m_{2x} - k = m_{2y}$ and $m_{nx} = m_{ny}$ are the existence conditions, and $\alpha > |m_{1y}| + |m_{2y}| + \cdots + |m_{ny}|$ is the sufficient stability condition.

5.CONCLUSIONS

In this paper we have studied the relation between parameter mismatches and synchronization in a certain class of unidirectionally linearly coupled time-delayed systems and have shown for the first time that parameter mismatches are of crucial importance for achieving synchronization. We have showed that independent of the relation between the feedback delay time in the coupled systems and the coupling delay time, only retarded (lag) synchronization with coupling delay lag time is obtained. We have established that either with parameter mismatch or without it neither complete nor anticipating chaos synchronization occurs. We have demonstrated our approach using the Ikeda and Mackey Glass models. We mention that, for example in the case of nonlinear (sinusoidal) coupling for identical drive and response Ikeda systems, depending on the relation between the feedback delay time and the coupling delay time retarded, complete or anticipating synchronization can occur [25]. These results are of significant interest in the context of relationship between parameter mismatches, coupling forms and synchronization. Indeed, having in mind possible practical applications of anticipating chaos synchronization [6] in secure communications (anticipation of the future states of the transmitter (master laser) at the receiver (slave laser) allows more time to decode the message), in the control of delay-induced instabilites in a wide range of non-linear systems, for the understanding of natural information processing choosing the "appropriate" parameters' mismatches and coupling forms certain types of synchronization can be switched off/on. We have also *for the first time* investigated chaos synchronization in variable delay time systems and found both existence and sufficient stability conditions for the retarded synchronization manifold with the coupling-delay lag time.

Also for the first time we have investigated synchronization between two unidirectionally coupled multi-feedback systems. These findings are of considerable interest in the context of synchronization between the stabilized laser systems (arrays) which hold great promise for space communication applications,where compact sources with high intensities are required. Additionally, synchronization between the multi-feedback systems can provide more flexibility in practical applications.

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